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That's a wrap!

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Abstract: Calibration technology provides us with a fast and elegant way to find the supergravity solutions for BPS wrapped M-branes. Its true potential had however remained untapped due to the absence of a classification of calibrations in spacetimes with non-trivial flux. The applications of this method were thus limited in practise to M-branes wrapping Kähler calibrated cycles. In this paper, we catagorize a type of generalised calibrations which exist in supergravity backgrounds and contain Kähler calibrations as a sub-class. This broadens the arena of brane configurations whose supergravity solutions are accessible through the calibration 'short-cut' method.

Keywords: M-Theory, p-branes.

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1. Introduction

Flat M-branes are solitons which preserve half the spacetime supersymmetry and are charged under the three-form of 11-dimensional supergravity. The BPS spectrum of M-Theory includes not just flat branes but also states corresponding to intersecting and wrapped brane configurations, the supergravity solutions of which have been widely discussed in recent years. (For reviews of the subject, see [1]). In spacetimes where the supergravity three-form is constant, supersymmetry preservation demands that compactification manifolds have special holonomy and the calibrations which exist on such manifolds are known. In more 'realistic' situations however, where the supergravity field strength is non-zero, we do not yet have an exhaustive list of possible calibrations, though relevant work has been done recently in [14] and [15].

Surfaces which M-branes wrap in order to produce BPS states can be classified using the mathematical theory of calibrations. In [8] it was shown how the supergravity solution for a wrapped M-brane follows almost immediately once we are given the p-form which calibrates the wrapped cycle. This procedure is both simple and elegant but since it could only be used if the calibrations were already known, its applications were limited in [8] to M-branes wrapped on Kähler calibrated surfaces in Kähler manifolds.

Using a different approach, the supergravity solution for an M5-brane on a Riemann surface in a three complex dimensional manifold was found in [7]. It turned out that the complex manifold need not be Kähler and is in fact defined through a somewhat unusual constraint on the hermitean metric. By investigating its implications for calibrations in this background, we are able to use the metric constraint to categorize a class of non-Kähler calibrations which exist in spacetimes with non-zero flux.

A complete classification of calibrations in such spacetimes will take substantial time and effort. We make a beginning here by studying the calibrations relevant to M-branes wrapping holomorphic cycles, as these are the simplest possible supersymmetric cycles one can consider. By exploiting the metric constraint referred to above, we are able to determine when non-Kähler calibrations arise and the conditions they are subject to.

We start in section 2 with a lightning review of BPS brane configurations in M-Theory. In section 3 we introduce calibrations in purely geometric backgrounds first before explaining the modifications necessary in order to extend the concept to include spacetime flux. In section 4 we outline first the method employed by Fayyazuddin and Smith [6] to find supergravity solutions for wrapped branes and then go on to show how the same results can be obtained more simply, a la [8], using calibrations.

In section 5 we present the main result of the paper: a general rule which enables us to determine the constraint on the metric (or calibration) corresponding to an M-brane wrapping a holomorphic cycle. This rule applies to both membranes and fivebranes, but since wrapped M2-branes have already been discussed in depth in [15], we concentrate mainly on M5-brane examples in section 6. Section 7 concludes with the reminder that though we have moved beyond Kählerity and catagorized a new type of calibration, many such classes still live in un-named oblivion, waiting to be found!

2. BPS branes in M-theory

2.1 Flat branes

We start by reviewing a few basic facts about M-branes. In the expressions which follow, as indeed in the rest of the paper, X^μ denotes coordinates tangent to a brane, X^α is used to denote transverse coordinates and $r = \sqrt{X^\alpha X_\alpha}$ is the radial coordinate in this transverse space.

The M5-brane. A flat M5-brane with worldvolume $X^{\mu_0} \dots X^{\mu_5}$ is a half-BPS object which preserves 16 real supersymmetries corresponding to the components of a spinor χ which satisfies the condition:

$$\hat{\Gamma}_{\mu_0\mu_1\mu_2\mu_3\mu_4\mu_5}\chi = \chi\,. \tag{2.1}$$

In the presence of this brane, the geometry of spacetime is described by the metric:

$$ds^2 = H^{-1/3}\eta_{\mu\nu}dX^\mu dX^\nu + H^{2/3}\delta_{\alpha\beta}dX^\alpha dX^\beta \tag{2.2}$$

where

$$H = 1 + \frac{a}{r^4}\,. \tag{2.3}$$

Since the M5-brane is charged under the supergravity three-form, it gives rise to a four-form field strength:

$$F_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta\rho}\partial_\rho H\,. \tag{2.4}$$

Together, the equations (2.3) and (2.4) specify the bosonic fields in the supergravity solution for the M5-brane.

The M2-brane. A flat M2-brane spanning directions $X^{\mu_0}X^{\mu_1}X^{\mu_2}$ is also a half-BPS object. Preserved supersymmetries correspond to the 16 components of a spinor χ which survive the following projection

$$\hat{\Gamma}_{\mu_0\mu_1\mu_2}\chi = \chi\,. \tag{2.5}$$

The bosonic content of the M2-brane supergravity solution is specified by the following expressions:

$$\begin{aligned} ds^2 &= H^{-2/3}\eta_{\mu\nu}dX^\mu dX^\nu + H^{1/3}\delta_{\alpha\beta}dX^\alpha dX^\beta & (2.6)\\ F_{\mu_0\mu_1\mu_2\alpha} &= \frac{\partial_\alpha H}{2H^2}\,, & (2.7)\end{aligned}$$

where

$$H = 1 + \frac{a}{r^6}\,. \tag{2.8}$$

2.2 Intersecting branes

One way of generating BPS states from the flat M-branes described above is to construct configurations of intersecting branes. In order for two M-branes to have a dynamic intersection, there must exist a worldvolume field to which the intersection can couple, either electrically or magnetically.

The bosonic scalars in the three-dimensional theory on the membrane worldvolume are dual to one-forms which can couple to point particles. Hence, a pair of membranes should overlap only at a point. Assuming that the self dual three-form on an M5 brane vanishes, the only bosonic fields turned on in the worldvolume theory of the fivebrane are scalars. Two M5-branes must intersect with each other in 3 spatial dimensions, in order for their intersection to couple (magnetically) to a scalar and thus have worldvolume dynamics. These arguments lead to what is known as the (p-2) self intersection rule [11] which states that BPS states can be built from M-branes if these are oriented such that each pair of Mp-branes has a $(p-2)$-dimensional spatial intersection .

The Killing spinors of the resulting intersecting brane system must simultaneously satisfy the projection conditions imposed by each of its flat M-brane constituents. In general, a complex structure can be defined on the relative transverse directions, i.e those which are spanned by some but not all of the constituent branes. The intersecting brane configuration can often be recovered as the singular limit of a single M-brane wrapping a smooth holomorphic cycle in this complex space.

2.3 Wrapped branes

Another way to generate BPS states is to wrap an M-brane on a supersymmetric n-cycle. It is known that holomorphic cycles are supersymmetric, and for the time being, we will restrict ourselves to considering these alone.

The amount of supersymmetry preserved by a particular spacetime can be found by introducing a brane which only probes the geometry and does not cause it to deformation in any way; Killing spinors of the probe brane are then Killing spinors of the background, as long as the probe is placed parallel to the wrapped brane and hence does not break any further supersymmetry. The Killing spinors of the probe brane satisfy the following projection condition [9]:

$$\chi = \frac{1}{p!}\epsilon^{\alpha_0\alpha_1\ldots\alpha_p}\Gamma_{M_0M_1\ldots M_p}\partial_{\alpha_0}X^{M_0}\partial_{\alpha_1}X^{M_1}\ldots\partial_{\alpha_p}X^{M_p}\chi \tag{2.9}$$

where α_i are worldvolume indices, and X^{M_i} describe the embedding of the p-brane in the ambient space-time. By virtue of being a spinor in 11 dimensions, χ must also be Majorana. This requirement, when imposed alongside (2.9) determines the number of supercharges preserved by the wrapped M-brane.

It is worth bearing in mind that intersecting brane configurations are a subclass of the wrapped branes considered in this paper and are obtained in the limit when some cycles the brane wraps become singular.

3. Calibrations and generalisations thereof

Calibrations ϕ_p are a very useful mathematical construction which enable us to classify minimal p-dimensional submanifolds in a given spacetime [2]. A closed p-form ϕ_p is called a **(standard) calibration** if it obeys the inequality

$$|\phi_{p_{|\Lambda_p}}| \leq dV_{\Lambda_p} \tag{3.1}$$

for all p-dimensional submanifolds Λ_p. A manifold Σ_p which saturates the inequality is known as a minimal or calibrated manifold.

In the absence of space-time flux, branes ensure their stability by minimizing their worldvolumes; the volume form of an uncharged stable brane must therefore be a calibration in spacetime[1]. In such purely geometric backgrounds, Killing spinors are covariantly constant.

If the brane is charged, we can take our pick of two alternate ways to describe the ambient spacetime. The bosonic fields needed to specify the background are the metric

[1]While the expressions we have used to illustrate these statements are taken from 11-dimensions, the same logic and results apply to lower dimensions as well.

and the flux of the gauge field which couples to the brane. The first and most obvious attitude we can take is to treat these fields as two seperate objects. Killing spinors of the brane configuration are determined by the metric as well as the field strength and are hence no longer covariantly constant. Since the coupling of the gauge potential to the brane must now be taken into account, it is only natural that the criterion for brane stability also should change. In fact it turns out that stability requires now that the energy of the brane be minimized, where the energy is a measure not only of the volume but also the charge.

Alternately, we could adopt a more unconventional point of view and 'combine' the effects of the metric and the flux into a suitably redefined metric. This new metric is defined in such a way that it sees Killing spinors as being covariantly constant. By shifting the effect of the field strength into the geometry, we have of course made the geometry substantially more complicated and the new metric will in general have torsion.

To reiterate then, the volume-form of a stable charged brane is minimal only when measured by a metric which has been defined so as to incorporate the effect of the flux. *Generalised calibrations* ϕ_p are thus defined such that

$$d(A_p + \phi_p) = 0 \tag{3.2}$$

and

$$|\phi_{p|_{\Sigma_p}}| \leq \tilde{dV}_{\Sigma_p} \tag{3.3}$$

for any p-dimensional submanifold Σ_p, where $\tilde{dV}$ is the volume form in terms of the redefined metric with torsion. It is also clear from the above that a generalised calibration is not closed, but rather is gauge equivalent to the potential A_p under which the $(p-1)$brane is charged.

Typically, only those directions along the brane worldvolume which are wrapped on a supersymmetric curve have a non-trivial space-time embedding; the remaining directions are flat[2]. Had its entire world-volume been flat, the brane would be a 1/2 BPS object; wrapped branes however, generically break more than half the supersymmetry. Since the preserved supercharges depend on the geometry of the supersymmetric cycle, it is the wrapped directions of the brane world-volume which play a essential role in this analysis whereas flat directions contribute trivially. Because of this, the volume form of a BPS brane, which is actually a calibration in the full spacetime, is sometimes referred to as a calibration even in the subspace where the cycle is embedded.

Clarifying our conventions. The definition of a generalised calibration in the embedding space $\mathcal{M}$ adopted here differs from that in [4]. In order to avoid any confusion in a comparison of the two papers, let us make this difference explicit.

A generalised calibration Φ_{p+1}in 11 dimensions is gauge equivalent to the potential A_{p+1} to which the Mp-brane couples electrically. We can split this up into the product of an l-form λ_l (where $l = p + 1 - m$) along the flat directions and an m-form ϕ_m in $\mathcal{M}$, such that both are generalised calibrations in their own subspaces. The components of these

[2] In the language of this paper, these are the X^μ

forms satisfy the relation:

$$\Phi_{i_0 \ldots i_{p+1}} = \lambda_{i_0 \ldots i_l} \times \phi_{i_{l+1} \ldots i_{p+1}} \equiv A_{i_0 \ldots i_{p+1}} \tag{3.4}$$

In [4], it is simply the gauge potential restricted to $\mathcal{M}$ which is called the generalised calibration $\tilde{A}_m$ in this subspace. While this is of course still an m-form, its components differ from those of ϕ_m, and can in fact be obtained from $A_{i_0 \ldots i_{p+1}}$ merely by holding the first l indices fixed, as follows:

$$\tilde{A}_{i_1 \ldots i_m} = A_{12 \ldots l, i_1 \ldots i_m} \,. \tag{3.5}$$

4. Supergravity solutions for wrapped branes

Supergravity solutions for intersecting brane configurations were constructed initially using the harmonic function rule [10], which is a recipe for combining the supergravity solutions of each component M-brane; the resulting solutions however were smeared along the relative transverse space. In [6], Fayyazuddin and Smith came up with a metric ansatz which enabled them to consider localised brane intersections. Since the method employed there will be used later in this paper, here is a brief overview of the way it works.

4.1 Fayyazuddin-Smith spacetimes

We begin by postulating a form for the spacetime metric, based on isometries of the wrapped brane configuration. Since we are focusing on bosonic backgrounds with Killing spinors, the gravitino has already been set to zero and supersymmetry preservation can be ensured simply by demanding that the gravitino variation vanish as well. This is imposed by hand and leads to a set of constraints on the field strength of the supergravity three form, as well as on the functions in the metric ansatz. If in addition, the field strength obeys the Bianchi Identity and equations of motion, it is guaranteed that Einstein's equations will be satisfied. Having outlined the method, we discuss a few of its steps in more detail in order to equip ourselves to apply the procedure to explicit examples.

Metric ansatz. We want to consider a class of spacetimes which describe M-branes wrapping supersymmetric cycles. Depending on the dimension of the cycle, there could be some world-volume directions X_μ which are left unwrapped or flat; Lorentz invariance should be preserved along these directions. If the supersymmetric cycle in question is embedded into an m dimensional subspace spanned by X_a, then the remaining directions of spacetime X_α, (those transverse to both X_μ and the embedding space), should be rotationally invariant. A metric describing this 11-dimensional spacetime takes the form:

$$ds^2 = {H_1}^2 \eta_{\mu\nu} dX^\mu dX^\nu + h_{ab} dX^a dX^b + {H_2}^2 \delta_{\alpha\beta} dX^\alpha dX^\beta \,. \tag{4.1}$$

From our discussion about the isometries of spacetime, we can see that the metric must be diagonal in both X_μ and X_α. However we cannot say anything about the metric in the embedding space yet beyond a comment that it must, along with H_1 and H_2, be independent of X_μ.

Since we are going to be concerned only with M-branes wrapping holomorphic cycles, it is convenient to define a complex structure on the embedding space. This makes the holomorphicity of the cycle (and therefore the supersymmetry of the configuration) manifest. We can hence replace the embedding space metric $h_{ab}dX^a dX^b$ with $2G_{M\bar{N}}dz^M dz^{\bar{N}}$, arriving at the following final form for the metric

$$ds^2 = {H_1}^2 \eta_{\mu\nu} dX^\mu dX^\nu + 2G_{M\bar{N}} dz^M dz^{\bar{N}} + {H_2}^2 \delta_{\alpha\beta} dX^\alpha dX^\beta \tag{4.2}$$

which will be used throughout this paper.

Killing spinors. A Majorana spinor in the 11-dimensional background described above can be decomposed into components along and transverse to the complex subspace where the supersymmetric cycle is embedded. The component which lies in the complex space can be expressed by a linear combination of Fock space states using the fact that the Clifford algebra in $\mathbb{C}^n$ takes on the form of an algebra of n fermionic creation and annihilation operators. The creation operators can then act on the vacuum to generate a Fock space in which states are labelled by n fermionic occupation numbers; one corresponding to each creation operator.

Supersymmetric variation of the gravitino. The Killing spinor for a particular brane configuration can be decomposed as described above and substituted in the gravitino variation equation:

$$\delta_\chi \Psi_I = \left(\partial_I + \frac{1}{4}\omega_I^{ij}\hat{\Gamma}_{ij} + \frac{1}{144}{\Gamma_I}^{JKLM}F_{JKLM} - \frac{1}{18}\Gamma^{JKL}F_{IJKL} \right)\chi\,. \tag{4.3}$$

The requirement $\delta_\chi \Psi = 0$ leads to the vanishing of a combination of Fock space states. Since these states are linearly independent, the coefficient of each must be set to zero identically. This gives us a set of constraints on the metric and field strength.

4.2 A shortcut ...

From our earlier discussion of calibrations, we have learnt that the generalised calibration ϕ corresponding to a wrapped brane is gauge equivalent to the $(p+1)$-form potential to which the brane couples. The field strength $F = dA = d\phi$, follows once we are given a suitable calibration. For the spacetimes described in the previous section, the calibrating form, and hence the field strength, can simply be read off from the metric! The entire bosonic content of the supergravity solution can thus be obtained simply by appealing to the isometries of the configuration and a knowledge of the possible calibrations.

In order to make this construction more explicit[3], we apply it to the example of a membrane wrapping a 2-cycle embedded holomorphically into $\mathbb{C}^3$.

M2 on a holomorphic curve in $\mathbb{C}^3$. The Fayyazuddin-Smith ansatz for the metric is:

$$ds^2 = -H^{2/3}dt^2 + 2G_{M\bar{N}}dz^M dz^{\bar{N}} + H^{1/3}\delta_{\alpha\beta}dX^\alpha dX^\beta\,, \tag{4.4}$$

[3] Details are given in [4] and [8].

where z^M are complex coordinates, X^α span the remaining 4 transverse directions and supersymmetry requirements have been used to fix two of the undetermined quantities in the metric ansatz in terms of a single harmonic function H. Given this metric, we can immediately read off the calibrating three-form

$$\Phi_{0M\bar{N}} = i\sqrt{H^{2/3}}dt \wedge G_{M\bar{N}}dz^M \wedge dz^{\bar{N}} \tag{4.5}$$

and use it to compute the components of the field strength:

$$F_{I0M\bar{N}} = d_{[I,}\Phi_{0M\bar{N}]} \tag{4.6}$$

where I can be any spatial coordinate.

In [8], where this method was proposed, it was assumed (for lack of any information to the contrary) that the metric $G_{M\bar{N}}$ is Kähler. However, as we shall soon see, this is by no means the only possibility!

4.3 . . . and its limitations!

Though calibration technology provides us with a procedure both quicker and simpler than the one described in section 4.1, the fact remains that in order to write down supergravity solutions for M-branes wrapped on calibrated manifolds, we need to know first what the calibrated forms are! Uptil now, this method had only been used for M-branes wrapped around Kähler calibrations [8] simply because these were the only calibrations we were aware of in spacetimes with non-trivial four-form flux, so the full potential which lay latent in this method could not be exploited.

This is a problem of larger proportions than one might at first think. While it is true that Kähler manifolds permeate String and M-Theory, it is also a fact that we frequently encounter non-Kähler manifolds, perhaps without recognizing them as such. In order to susbtantiate this statement, we take a short detour through a brane described by a non-Kähler calibration, before proceeding to discuss the general constraint which can be used to identify a class of calibrations in M-Theory backgrounds.

Non-Kähler calibrations. A flat M5-brane can be thought of as being wrapped on a trivial supersymmetric cycle embedded into a subspace of 11 dimensional spacetime. Take this subspace to be $\mathbb{C}^3$, spanned by holomorphic coordinates u, v, w. Two equations $f = f(u, v, w) = 0$ and $g = g(u, v, w) = 0$ are then needed to define a holomorphic two-cycle. If these equations are $v = 0$ and $w = 0$, the two-cycle in question is simply the complex u plane.

In the presence of a flat M5-brane with worldvolume $0123u\bar{u}$, the spacetime metric is given by:

$$ds^2 = H^{-1/3}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dud\bar{u}\right) + H^{2/3}\left(dvd\bar{v} + dwd\bar{w} + dy^2\right)$$

where

$$H = \frac{\text{constant}}{(|v|^2 + |w|^2 + y^2)^{3/2}}\,.$$

In general, an M5-brane wrapping a Riemann surface embedded in $\mathbb{C}^3$ is expected to give rise to a metric of the form [7]:

$$ds^2 = H^{-1/3}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + 2G_{M\bar{N}}dz^M dz^{\bar{N}} + H^{2/3}dy^2$$

where 0123 are the flat directions of the brane and y is the single coordinate transverse to both the brane and the complex space. Comparing the two expressions, we can read off the components of the hermitean metric:

$$2G_{u\bar{u}} = H^{-1/3}\,, \qquad 2G_{v\bar{v}} = H^{2/3}\,, \qquad 2G_{w\bar{w}} = H^{2/3}\,.$$

It is convenient to define a rescaled metric, $g_{M\bar{N}} = H^{-1/6}G_{M\bar{N}}$ whose components are

$$2g_{u\bar{u}} = H^{-1/2}\,, \qquad 2g_{v\bar{v}} = H^{1/2}\,, \qquad 2g_{w\bar{w}} = H^{1/2}\,.$$

It can trivially be seen that this metric is not Kähler; moreover since a Kähler metric cannot be obtained even by rescaling, $g_{M\bar{N}}$ is not warped Kähler either. However, the components of this blatantly non-Kähler metric satisfy the following curious relations:

$$\partial_u g_{v\bar{v}} g_{w\bar{w}} = \partial_v g_{u\bar{u}} g_{w\bar{w}} = \partial_w g_{u\bar{u}} g_{v\bar{v}} = 0\,. \tag{4.7}$$

In terms of the hermitean form $\omega = i g_{M\bar{N}} dz^M dz^{\bar{N}}$ associated with the metric, this can be re-expressed as follows:

$$\partial[\omega \wedge \omega] = 0 \quad \text{but} \quad \partial\omega\,. \neq 0$$

As we will see in the following sections, this M5-brane is by no means the sole example of commonly encountered non-Kähler calibrations!

5. A method to the madness

In our search for non-Kähler calibrations, we don't exactly have to look very far. Apart from the M5 example quoted above, there are membranes aplenty willing to oblige!

5.1 The madness

Supergravity solutions for a class of BPS states corresponding to wrapped membranes were dicussed in [15]. In keeping with the logic that holomorphicity implies supersymmetry, the M2-branes were wrapped on holomorphic cycles in complex subspaces of varying dimension n. Working with the following (standard) ansatz for the spacetime metric:

$$ds^2 = -H^{-2/3}dt^2 + 2G_{M\bar{N}}dz^M dz^{\bar{N}} + H^{1/3}\delta_{\alpha\beta}dX^\alpha dX^\beta\,, \tag{5.1}$$

where z^M are n holomorphic coordinates and X^α span the $(10-2n)$ transverse directions, it was found that in each case, supersymmetry preservation imposes a constraint on the hermitean metric. These constraints take the following form:

$$\begin{aligned}
\partial[H^{2/3}\omega] &= 0 \quad \text{for}\, n=2 \\
\partial[H^{1/3}\omega\wedge\omega] &= 0 \quad \text{for}\, n=3 \\
\partial[\omega\wedge\omega\wedge\omega] &= 0 \quad \text{for}\, n=4 \\
\partial[H^{-1/3}\omega\wedge\omega\wedge\omega\wedge\omega] &= 0 \quad \text{for}\, n=5
\end{aligned} \tag{5.2}$$

where $\omega = iG_{M\bar{N}}dz^M \wedge dz^{\bar{N}}$ is the hermitean two-form associated with the metric in the complex subspace.

While (warped) Kähler metrics definitely solve the above constraints they by no means exhaust the available options. Hence, by assuming that the metric on the embedding space is Kähler, we are in fact restricting ourselves unnecesarily and losing out on a wealth of possibilities.

We have now discussed two ways of obtaining supergravity solutions. Each has its merits. The first, explained in section 4.1, involves somewhat lengthy computations but includes in its results a statement about the kind of manifold the embedding space must be. The second procedure, outlined in section 4.2, is quick and simple but it does not yield a restriction on the embedding space metric; this must be provided as an external input.

Poised between the labour of one method, and the simplicity of the other, it is inevitable to ask the question, can we somehow combine the best features of both approaches? Is it possible to broaden the class of calibrations under consideration so that we have the ease of writing down supergravity solutions using calibrations and also the assurance that all possible cases are covered?

5.2 The method

A persistent feature of the wrapped brane supergravity solutions obtained using the method of [6] is a constraint on the metric in the subspace where the supersymmetric cycle is embedded. This constraint in turn restricts the $(p+1)$-form potential to which the brane couples. Since the potential is gauge equivalent to the generalised calibration, we find that in fact the metric constraint can alternately be viewed as a condition which determines generalised calibrations in the given background.

We propose the following: *in 11-dimensional backgrounds with non-zero four-form flux, a class of generalised calibrations in the embedding space M is given by the $2m$-forms ϕ_{2m}, for*

$$\phi_{2m} = (\underbrace{\omega \wedge \omega \wedge \cdots \omega \wedge \omega}_{m}) \tag{5.3}$$

if the following constraint holds:

$$\partial *_{\mathcal{M}} \left[\phi_{2m} |G^{'}|^{1/2m}\right] = 0\,. \tag{5.4}$$

Here, $G^{'}$ denotes the determinant of the metric restricted to directions transverse to the embedding space, and the Hodge dual is taken within the embedding space.

6. Proof by example

The aim of this section is to illustrate the exhaustive nature of the statement (5.4) through a case by case analysis of M-branes wrapping holomorphic cycles in complex manifolds of various dimensions.

6.1 M2-branes on 2-cycles

Two cycles are the only holomorphic curves on which M2-branes can be wrapped. In an 11-dimensional spacetime, these curves can be embedded in manifolds of complex dimensions 23, 4 and 5. The supergravity solutions corresponding to each of these configurations were studied in [15] and the metric constraints that accompany each solution were reviewed in (5.2). Notice now that they can be unified into the following expression:

$$\partial *_{\mathbb{C}} \left[\omega_{M\bar{N}}\sqrt{\det G'}\right] = 0 \tag{6.1}$$

where $\det G'$ is the determinant of the metric when restricted to directions transverse to the complex sub-manifold.

The part of the gauge potential to which the membrane worldspace couples is given simply by $\omega_{M\bar{N}}$. In order for the membrane to be stable and supersymmetric, this two form should be a generalised calibration in the complex space. Hence, we find that the constraint (6.1) is but a special case of the general rule (5.4) as applied to M2-branes.

6.2 M5-branes on 2-cycles

In this section we employ the calibration approach to find supergravity solutions for five-branes wrapped on holomorphic two-cycles, using calibrations determined by the constraint (5.4). This procedure enables us to reproduce in a few simple steps the results of [6] and [7].

M5 on a holomorphic curve in $\mathbb{C}^2$. When an M5-brane wraps a holomorphic sub-manifold in $\mathbb{C}^2$, the relevant ansatz for the spacetime metric is:

$$ds^2 = H^{-1/3}\eta_{\mu\nu}dX^\mu dX^\nu + 2G_{M\bar{N}}dz^M dz^{\bar{N}} + H^{2/3}\delta_{\alpha\beta}dX^\alpha dX^\beta\,. \tag{6.2}$$

where z^M are coordinates on $\mathbb{C}^2$, α takes values $8, 9, 10$ and μ runs over $0, 1, 2, 3$. Supersymmetry preservation has already been used to fix the relative coefficients $H^{-1/3}$ and $H^{2/3}$. It further dictates that, (upto rescaling by an arbitrary holomorphic function), the harmonic function H is related to the determinant G of the Hermitan metric by

$$\sqrt{G} = H^{2/3}\,.$$

The calibrating form of the M5-brane

$$\Phi = H^{-2/3}G_{M\bar{N}}dt\wedge dX^1\wedge dX^2\wedge dX^3\wedge dz\wedge dz^{\bar{N}} \tag{6.3}$$

$$= dV_{0123}\wedge\phi_{M\bar{N}} \tag{6.4}$$

must be such that:

$$\partial\left[H^{1/3}G_{M\bar{N}}\right] = 0\,. \tag{6.5}$$

If the above condition is satisfied, the generalised calibration is gauge equivalent to the potential A under which the brane is charged. The only non-vanishing component of this potential can then be read off from (6.4) as:

$$A_{0123M\bar{N}} = H^{-2/3}G_{M\bar{N}}\,. \tag{6.6}$$

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Consequently, the supergravity four form $F_4 = *dA$ is given by:

$$F_{M89(10)} = -\frac{i}{2}\partial_M H \tag{6.7}$$

$$F_{\bar{N}89(10)} = \frac{i}{2}\partial_{\bar{N}} H \tag{6.8}$$

$$F_{N\bar{M}\beta\gamma} = \frac{i}{2}\epsilon_{\alpha\beta\gamma}\partial_\alpha \left[H^{1/3}G_{N\bar{M}}\right]. \tag{6.9}$$

These results agree exactly with those obtained in [6].

M5 on a holomorphic curve in $\mathbb{C}^3$. When the M5brane is wrapped on a holomorphic curve embedded in $\mathbb{C}^3$, the metric takes the form:

$$ds^2 = H^{-1/3}\eta_{\mu\nu}dX^\mu dX^\nu + G_{M\bar{N}}dz^M dz^{\bar{N}} + H^{2/3}dy^2 . \tag{6.10}$$

where z^M now span $\mathbb{C}^3$, y is the single overall transverse direction. and the harmonic function H is related to the determinant of the hermitean metric by $H = \sqrt{G}$. In this background, the wrapped M5-brane is calibrated by the form

$$\Phi = H^{-2/3}G_{M\bar{N}}dt \wedge dX^1 \wedge dX^2 \wedge dX^3 \wedge dz \wedge dz^{\bar{N}} \tag{6.11}$$

$$= dV_{0123} \wedge \phi_{M\bar{N}} \tag{6.12}$$

such that

$$\partial \left[H^{-1/3}\omega_G \wedge \omega_G\right] = 0 . \tag{6.13}$$

When this relation holds, the gauge potential A is once more given by

$$A_{0123M\bar{N}} = H^{-2/3}G_{M\bar{N}} . \tag{6.14}$$

This leads to the following expressions for the four-form field strength:

$$F_{\bar{N}\bar{P}My} = \frac{1}{2}\left[\partial_{\bar{P}}\left(H^{1/3}G_{M\bar{N}}\right) - \partial_{\bar{N}}\left(H^{1/3}G_{M\bar{P}}\right)\right], \tag{6.15}$$

$$F_{NP\bar{M}y} = \frac{1}{2}\left[\partial_P\left(H^{1/3}G_{N\bar{M}}\right) - \partial_N\left(H^{1/3}G_{P\bar{M}}\right)\right], \tag{6.16}$$

$$F_{MN\bar{P}\bar{Q}} = \frac{i}{2}\partial_y\left[H^{-1/3}\left(G_{M\bar{Q}}G_{N\bar{P}} - G_{M\bar{P}}G_{N\bar{Q}}\right)\right] \tag{6.17}$$

which are precisely the expressions obtained in [7].

6.3 M5-branes on 4-cycles

The worldvolumes of fivebranes are large enough to be wrapped on holomorphic four-cycles as well. The smallest complex space into which a four-manifold can have a non-trivial holomorphic embedding is $\mathbb{C}^3$, and this is the configuration we turn to now. As we will see later, this is in fact the only M-brane which can be wrapped on a holomorphic four-cycle within the scope of this paper.

The Fayyazuddin-Smith treatment.

Metric ansatz: For an M5-brane wrapped on a 4-cycle Σ_4 in $\mathbb{C}^3$, the metric takes the form:

$$ds^2 = H_1^2 \eta_{\mu\nu} dX^\mu dX^\nu + 2G_{M\bar{N}} dz^M dz^{\bar{N}} + H_2^2 \delta_{\alpha\beta} dX^\alpha dX^\beta \tag{6.18}$$

where $\mu = 0, 1$ labels the unwrapped directions, z^M are holomorphic coordinates in $\mathbb{C}^3$ and α takes values 8,9, and 10.

Killing Spinors: The Killing spinors in this space-time are given by (2.9):

$$\epsilon^{abcd} \Gamma_{01} \Gamma_{m\bar{n}p\bar{q}} \partial_a X^m \partial_b X^{\bar{n}} \partial_c X^p \partial_d X^{\bar{q}} \chi = \chi \tag{6.19}$$

where the Γ_m are flat space Γ-matrices and $\sigma^a \ldots \sigma^d$ are coordinates on the four-cycle. This leads to the condition

$$\Gamma_{01} \Gamma_{m\bar{n}p\bar{q}} \chi = (\eta_{m\bar{n}} \eta_{p\bar{q}} - \eta_{m\bar{q}} \eta_{p\bar{n}}) \chi \tag{6.20}$$

where $\eta_{m\bar{n}}$ is the flat space metric. A solution is given by

$$\chi = a \otimes |000\rangle + b \otimes |111\rangle \tag{6.21}$$

if the spinors a and b in the $(1 + 4)$ dimensional space-time transverse to $\mathbb{C}^3$ satisfy:

$$\begin{aligned} \Gamma_{89(10)}\, a &= -ia \\ \Gamma_{89(10)}\, b &= ib\,. \end{aligned} \tag{6.22}$$

The wrapped M5-brane then preserves $\frac{1}{8}$ of the spacetime supersymmetry, corresponding to the 4 spinors which satisfy the above conditions.

Supersymmetric variation of the gravitino: The BPS supergravity solutions for the wrapped brane are found by demanding that $\delta_\chi \Psi = 0$ holds for a metric of the form (4.2), when the variation parameter χ is given by (6.21). This gives rise to the following set of equations:

$$3\partial_{\bar{M}} ln\ H_1 - iH_2^{-3} F_{\bar{M}89(10)} = 0 \tag{6.23}$$

$$3\hat{\epsilon}_{\alpha\beta\gamma}(\partial_\gamma H_2) + 2iG^{M\bar{N}} F_{M\bar{N}\beta\gamma} = 0 \tag{6.24}$$

$$3G^{N\bar{P}}(\partial_{\bar{P}} G_{N\bar{M}}) + 4iH_2^{-3} F_{\bar{M}89(10)} = 0 \tag{6.25}$$

$$6iH_2 \partial_\alpha ln\ H_1 + \hat{\epsilon}_{\alpha\beta\gamma} G^{M\bar{N}} F_{M\bar{N}\beta\gamma} = 0 \tag{6.26}$$

$$iH_2^{-1}[6F_{M\bar{N}\beta\gamma} - 2G_{M\bar{N}} G^{P\bar{Q}} F_{P\bar{Q}\beta\gamma}] + 3\hat{\epsilon}_{\alpha\beta\gamma}(\partial_\alpha G_{M\bar{N}}) = 0 \tag{6.27}$$

$$3H_2^3(\partial_{\bar{P}} G_{M\bar{N}} - \partial_{\bar{N}} G_{M\bar{P}}) - 2iG_{M\bar{N}} F_{\bar{P}89(10)} + 2iG_{M\bar{P}} F_{\bar{N}89(10)} = 0\,. \tag{6.28}$$

The orginal variables: an aside. Formulating the solutions in terms of H_1, H_2 and $G_{M\bar{P}}$, we find that the harmonic functions, though related to each other as expected, are independent of the determinant, G. Moreover,

$$\partial_\alpha G = \partial_M G = \partial_{\bar{N}} G = 0\,.$$

Though this might seem puzzling at first, it is important to realise that we are not implying that the hermitean metric is independent of the spatial coordinates; merely that such dependences cancel out in its determinant! That this is infact to be expected can be seen by looking at a flat M5 brane, trivially embedded in $\mathbb{C}^3$.

Example: For an M5 spanning $01u\bar{u}v\bar{v}$, the supergravity solution is given by

$$ds^2 = H^{-1/3}\left(-dt^2 + dx_1^2 + dud\bar{u} + dvd\bar{v}\right) + H^{2/3}(dwd\bar{w} + dx_7^2 + dx_8^2 + dx_9^2 + dx_{10}^2),$$

enabling us to read off the following:

$$H_1^2 = H^{-1/3}\,, \quad H_2^2 = H^{2/3} \;\Rightarrow\; H_1^2 = H_2^{-1} \tag{6.29}$$

and

$$2G_{u\bar{u}} = 2G_{v\bar{v}} = H^{-1/3}\,, \quad 2G_{w\bar{w}} = H^{2/3} \;\Rightarrow\; \sqrt{G} = G_{u\bar{u}}G_{v\bar{v}}G_{w\bar{w}} = 1/8\,. \tag{6.30}$$

Since the determinant of the hermitean metric is a constant, its derivatives obviously vanish.

In terms of the rescaled quantities:

$$H \equiv H_2^3 \quad \text{and} \quad g_{M\bar{N}} \equiv H_2 G_{M\bar{N}} \tag{6.31}$$

solutions to the above equations are:

$$\partial_{\bar{N}}\, lnH \;=\; -6\partial_{\bar{N}}\, lnH_1 = 3\partial_{\bar{N}}\, lnH_2 \tag{6.32}$$

$$\partial_{\bar{P}} g_{M\bar{N}} \;=\; \partial_{\bar{N}} g_{M\bar{P}} \tag{6.33}$$

$$F_{89(10)M} \;=\; \frac{i}{2}\partial_M H \tag{6.34}$$

$$F_{N\bar{M}\beta\gamma} \;=\; \frac{i}{2}\epsilon_{\alpha\beta\gamma}\partial_\alpha g_{N\bar{M}} \tag{6.35}$$

Since the supergravity three-form couples magnetically to the M5-brane, $d * F = 0$ trivially, whereas the Bianchi Identity constrains the metric to obey the following non-linear differential equation

$$\partial_\alpha^2 g_{M\bar{N}} + 2\,\partial_M\partial_{\bar{N}} H = 0 \tag{6.36}$$

This is precisely the solution found in [8]. What we have gained through the above analysis is the metric constraint which explicitly rules out the possibility of M5-branes being wrapped on non-Kähler holomorphic four-cycles in three complex dimensional manifolds and tells us that the previously calculated solution [8] is, in this case, the only option!

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The calibration method. The only non-vanishing component of the gauge potential corresponding to an M5-brane wrapping a holomorphic four-cycle in $\mathbb{C}^3$ is

$$\tilde{A}_{01MN\bar{P}\bar{Q}} = H^{-1/3}(G_{M\bar{P}}G_{N\bar{Q}} - G_{M\bar{Q}}G_{N\bar{P}})\,. \tag{6.37}$$

Decomposing this into a two-form dV_2 along the $(0,1)$ directions and a four-form $\phi_{MN\bar{P}\bar{Q}}$ in $\mathbb{C}^3$, we find that the gauge-potential/world-volume is a calibration in space-time only if $\phi_{MN\bar{P}\bar{Q}}$ is a generalised calibration in the embedding space, i.e. it satisfies (5.4). For an M5-brane wrapped on a holomorphic 4-cycle in $\mathbb{C}^3$, this imples that

$$\partial *_{\mathbb{C}} \left[H^{1/3}\omega_G \wedge \omega_G\right] = \partial\omega_g = 0 \tag{6.38}$$

where ω_G and ω_g are the hermitean two-forms associated with the metrics $G_{M\bar{N}}$ and $g_{M\bar{N}}$ respectively. The field strength of the supergravity three form can now be easily calculated using $F_4 = *dA_6$, reproducing previous results.

7. The odd one out *or* what lies ahead

In order for a wrapped M-brane to be a stable, supersymmetric configuration the gauge potential to which it couples must be (equivalent to) a generalised calibration in 11-dimensional space-time. In this paper, we looked at M-branes wrapping holomorphic curves and found that when the background contains a field strength, supersymmetric cycles/calibrated forms in a complex subspace have a non-trivial dependence on the remaining directions of spacetime, as reflected in the constraint (5.4).

The examples considered here, together with those in [15] exhaust all possible cases of M-branes wrapped holomorphic cycles in complex subspaces of 11-dimensional space-time; with one notable exception. An M5 wrapped on a four-cycle in $\mathbb{C}^4$ has not been discussed and in fact its supergravity solution cannot be constructed based on a calibration following from (5.4). We justify its exclusion from the present analysis by pointing out how it differs from the configurations we have looked at so far.

The definition of generalised calibrations presented in section 3, and as a result the entire formalism of this paper, is restricted by construction to apply to configurations with a non-vanishing spacetime flux and no non-trivial fields on the brane world-volumes [4]. The rule (5.4) comes with the in-built assumption that the only bosonic fields turned on in the M-brane worldvolume theory are scalars. As a result, it applies to *only* to those wrapped M-branes which, in the intersecting brane picture can be interpreted as a system of branes where each pair has a $(p-2)$ dimensional spatial overlap.

It should by now be clear why (5.4) cannot be applied to an M5 wrapping a four-cycle in $\mathbb{C}^4$. In the intersecting brane picture obtained by taking the limit where the four-cycle becomes singular, the two M5-branes have only one common spatial dimension, thus violating the $(p-2)$ self intersection rule!

[4] A suitable modification of the calibration concept does exist which incorporates worldvolume fluxes [13], however it has not been undertaken here.

Though, by considering a particular type of BPS wrapped M-branes, we have succeeded in shedding light on a new catagory of calibrations, many still lurk in the shadows. This message is reinforced by the example discussed above. It is hoped that a study of this brane configuration will bring another class of calibrations into the spotlight. This theme will be expanded upon further in a paper coming soon to an arXiv near you!

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